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$$9^h\ \ 57^m\ \ 1^s.91$$

as determined by Mr. Airy. But Mr. Airy did not determine the rotation of Saturn but of Jupiter, and the time given is the *siderial* time of the rotation of Jupiter, which still needs a small correction on account of aberration. 1877, Jan. 20.

CLASSIFICATION OF PLANE CURVES WITH REFERENCE TO INVERSION.

BY PROF. W. W. JOHNSON, ST. JOHN'S COLLEGE, ANNAPOLIS, MD.

1. THE degree of the inverse of a given curve of the nth degree is in general $2n$, (thus the inverse of a conic is in general of the 4th degree,) nevertheless it is possible so to classify curves that a given curve and its inverse with reference to any point shall belong to the same family.

The polar coordinates $(r'\ \theta')$ of the point on the inverse curve corresponding to the point $(r,\ \theta)$ of the given curve are $r' = k^2 \div r,\ \theta' = \theta$; where k^2 is the modulus of inversion, and the centre of inversion is the pole. Hence denoting the rectangular coordinates by $(x,\ y)$ and $(x',\ y')$ we derive

$$x = \frac{k^2x'}{r'^2} \quad \text{and} \quad y = \frac{k^2y'}{r'^2}.$$

Substituting these values for x and y in the equation of the given curve we have the equation of the inverse curve, r'^2 being finally replaced by $x'^2+y'^2$.

2. Now the highest terms of the given equation being denoted by u_n, those of the $(n-1)$th degree by u_{n-1} &c.; if u_n contains a power of x^2+y^2 as a factor, let this factor be replaced by the like power of r^2. Let the terms u_{n-1} be treated in the same way, and so on with each group of terms of a degree exceeding $\frac{1}{2}n$. Regarding r^2 for a moment as a factor of only *one dimension*, ascertain the degree of the equation in r^2, x and y. Call this the *circular degree* of the equation, and denote it by m. The complete equation of the mth circular degree may then be written in the annexed tabular form. The terms of like degree are written in the same horizontal line and the degree of the eq'n cannot exceed $2m$. The *circular degree* of all the terms above the central line is m.

$$\left.\begin{array}{c} Ar^{2m} \\ +(Bx+Cy)r^{2(m-1)} \\ +(Dx^2+Exy+Fy^2)r^{2(m-2)} \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ Mx^m+Nx^{m-1}y+\ \ldots\ldots\ldots\ +Ty^m \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ +\ D'x^2+\ E'xy+\ F'y^2 \\ +\ B'x+\ C'y \\ +A' \end{array}\right\} = 0$$

3. *The circular degree of an equation is not changed by transformation of coordinates*; for, when the direction of the axes is changed without change of origin, r^2 is unchanged, while for x and for y linear functions of x and y are substituted; and when the origin is transferred say to the point (a, b), we have to substitute for x, $x+a$, for y, $y+b$ and for r^2 the expression $r^2 + 2ax + 2by + a^2 + b^2$, all of which are of the same circular degree with the expressions they replace.

We may therefore classify curves into *circular orders* according to the circular degrees of their equations. It is obvious that if $S = 0$ is the equation of a curve of the pth circular order, and $S' = 0$ that of a curve of the qth circular order, the circular order of $SS' = 0$ will be $p+q$, and that of $S+kS' = 0$ will be p, supposing q to be not greater than p.

4. *A curve and its inverse are of the same circular order.* The origin of co-ordinates being taken at the required centre of inversion we obtain the circular equation of the inverse by making in the given equation the substitutions,

$$x = \frac{k^2 x'}{r'^2}, \qquad y = \frac{k^2 y'}{r'^2}, \qquad r^2 = \frac{k^4}{r'^2}.$$

Let

$$A x^\alpha y^\beta (r^2)^p \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (1)$$

represent any term of the given equation of the mth circular degree. The circular degree of this term, namely $\alpha+\beta+p$, cannot exceed m; then let

$$\alpha+\beta+p+q = m.$$

By substitution, the term (1) becomes

$$A \frac{x'^\alpha y'^\beta (k^2)^{\alpha+\beta+2p}}{(r'^2)^{\alpha+\beta+p}}.$$

There is at least one term in the given equation, in which $\alpha + \beta + p = m$, hence, to free the equation of the inverse from fractions, it is always necessary to multiply throughout by $(r'^2)^m$; the above term of the inverse then becomes, (omitting the accents),

$$A x^\alpha y^\beta (r^2)^q (k^2)^{\alpha+\beta+2p}. \quad \ldots\ldots\ldots\ldots\ldots \quad (2)$$

The circular degree of this term is $\alpha + \beta + q$; and, since $p = 0$ in at least one of the terms of the original equation, (otherwise it would have been divisible by r^2), the greatest value of $\alpha + \beta + q$ is m, hence the inverse curve is of the mth circular order.

5. It will be observed that the exponents of x and y are unchanged, and that the sum of the indices of the degree (not circular) of the terms (1) and (2) is $2m$, hence we may say that each term is converted into one of *complementary degree*, with respect io $2m$, this conversion being effected by

simply adjusting the exponent of r^2, and then multiplying by the power of the modulus indicated by the degree $(\alpha+\beta+2p)$ of the original term.

Thus the equation of the cissoid is

$$r^2x = 2ay^2 \tag{3}$$

in which $m = 2$: taking $k = 2a$, the inverse is

$$k^6x = 2ay^2k^4$$

or the parbola

$$y^2 = 2ax. \tag{4}$$

The equation of the equilateral hyperbola is

$$x^2 - y^2 = a^2. \tag{5}$$

taking $k = a$, the inverse is the lemniscate,

$$r^4 = (x^2+y^2)^2 = a^2(x^2-y^2). \tag{6}$$

Again, in the case of the cissoid, if we transfer the origin to the point $(a, 0)$ [centre of the generating circle] we have the equation

$$r^2x + 3ax^2 - ay^2 + 3a^2x + a^3 = 0, \tag{7}$$

hence the inverse with respect to this point, and modulus a^2, is

$$r^4 + 3r^2ax + 3a^2x^2 - a^2y^2 + a^3x = 0. \tag{8}$$

6. In the general equation of the mth circular degree, as arranged in Art. 2, the complementary terms are those *symmetrically situated with respect to the central row of terms;* hence if we make $k^2 = 1$, the equation of the inverse curve will be found by simply interchanging A and A', B and B', &c.

7. The general equation of the mth *circular* degree includes every equation of the mth degree in which the group of terms of highest degree is not divisible by x^2+y^2. If these terms are so divisible the equation belongs to a lower circular order. Thus the second circular order includes all conics except the circle, which together with the straight line constitutes the first circular order.

8. Denoting the circular order of the curve by m, and the degree of the equation by n, if $n = m + 1$, so that the highest terms of the equation are in the first row above the central row, the curve is said to be a circular $(m+1)$–ic. If $n = m + 2$ the curve is said to be a bi-circular $(m + 2)$–ic. Thus the second circular order includes besides the non-circular conics, the circular cubics and the bi-circular quartics. In general, the mth circular order embraces the q-circular $(m + q)$–ics, where q may be called the index of circularity, and the degree of the curve is

$$n = m + q,$$

in which q cannot exceed m. Hence, among curves of a given degree, the curves of highest circularity belong to the lowest circular order.

9. The points at infinity in a given curve are found by equating to zero the terms of highest degree; hence to determine these points for the complete equation which represents an m-circular $2m$-ic, we have

$$Ar^{2m} = A(x^2+y^2)^m = A(x+iy)^m(x-iy)^m = 0,$$

where $i = \sqrt{-1}$. Hence the only points at infinity are the imaginary circular points, and the curve meets the line at infinity m times at each of these points.

The curve has m parallel asymptotes passing through each of these points: for assume the equation

$$x + iy = \beta,$$

and eliminate y from the equation of the curve by means of this equation. We have

$$r^2 = 2\beta x - \beta^2,$$

hence the resulting equation will reduce to one of the mth degree in x, and the coefficient of x^m will obviously contain β^m. It follows that m values of β may be determined, such that the line $x + iy = \beta$ meets the curve in one more point at infinity; in other words, we may determine m parallel asymptotes. The imaginary point at infinity determined by this equation is therefore said to be a *multiple point of the order m.*[*] In like manner the other circular point is a multiple point of the same order.

10. If in the general equation $A = 0$, $q = m - 1$, and the terms of highest degree are

$$(Bx + Cy)(x^2 + y^2)^{m-1}.$$

The circular points at infinity may now be shown to be multiple points of the order $m - 1$, and there is one real point at infinity determined by the equation

$$Bx + Cy = 0.$$

In this case, according to Art. 6, the absolute term will vanish from the equation of the inverse curve; hence this curve will pass through the centre of inversion; and, since the terms of the lowest degree in its equation will, by Art. 5, be $(k^2)^{2m-1}(Bx+Cy)$, the above equation will also represent the tangent to the inverse curve at the centre of inversion. It is indeed evident geometrically that a branch passing to infinity in any direction inverts into a branch passing in that direction through the centre of inversion. Thus the cissoid, in whose equation, (3) or (7) Art. 5, the term containing r^4 is wanting, is a circular cubic having its real point at infinity in the direction of the axis of y; accordingly each of the inverse curves, (4) and (8), touches the axis of y at the centre of inversion.

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[*] If the terms of lower degree were not divisible by the powers of r^2 as indicated in the general equation (Art. 2), the circular points would *not* be points of the mth order of multiplicity. For instance, if the highest terms of a quartic were $(x^2+y^2)^2$, and the cubic terms did not contain (x^2+y^2) as a factor, the result of elimination would be of the third degree, and β would not enter the coefficient of x^3; hence no asymptotes could be determined. The circular points would not, in that case, be nodes, but points where the curve *touches* the line at infinity; analogous to the real points corresponding to parabolic branches. The curve instanced would be a case of the circular, not of the bi-circular quartic.

11. If in the general equation $A = 0$, $B = 0$ and $C = 0$, we have $q = m - 2$; the terms of highest degree then are

$$(Dx^2 + Exy + Fy^2)(x^2 + y^2)^{m-2}.$$

The circular points are now multiple points of the order $m - 2$, and there are two other points at infinity determined by the eqnation

$$Dx^2 + Exy + Fy^2 = 0.$$

The terms of lowest degree in the equation of the inverse curve will be $(k^2)^{2m-2}(Dx^2 + Exy + Fy^2)$, hence that curve will have a node at the centre of inversion, and the above equation determines also the tangents to the inverse curve at that point. Thus in the case of the hyperbola, equ'n (5), which is a non-circular conic [$m = 2$, $q = 0$], with real points at infinity in the direction $y = \pm x$, the inverse curve (6) is a bi-circular quartic with a double point at the centre of inversion, at which point the tangents are $y = \pm x$. When, as in this example, the points at infinity are real and different the inverse has a crunode. When, as in the case of the parabola, eq. (4), the points at infinity are coincident, the inverse has a cusp at the centre of inversion like the cissoid of eq. (3). Finally, when the points at infinity are imaginary, the inverse has imaginary branches passing through the centre of inversion, which is therefore an acnode.

12. In general, q being the index of circularity, the terms of highest degree are of the form

$$(Hx^{m-q} + Ix^{m-q-1}y \dots .)(x^2 + y^2)^q$$

the circular points at infinity are multiple points of the order q, and there are m—q other points at infinity. The inverse curve has a multiple point at the centre of inversion of the order m—q. Thus all the points at infinity of the given curve are represented by real or imaginary branches of the inverse curve passing through the centre, except the $2q$ points which coincide with the circular points at infinity.

13. Conversely, the given curve being a q-circular n-ic, $m = n - q$. If the centre of inversion is not a point of the curve, the inverse is an m-circular $2m$-ic. If the centre of inversion is an ordinary point of the curve, the inverse is an (m—1)-circular (2m—1)-ic, and has one real point at infinity.

If the centre of inversion is a node, the inverse is an $(m-2)$-circular $(2m-2)$-ic, having two real, coincident or imaginary points at infinity, according as the node is a crunode, cusp or acnode. In general, if the centre of inversion is a multiple point of the s order, the inverse is an $(m-s)$-circular $(2m-s)$-ic, with s real or imaginary points at infinity corresponding to the s branches of the given curve at the centre of inversion.

14. It is evident geometrically that a node or cusp, *not* at the centre of inversion, inverts into a similar point, so that the possession of these singularities affords a sub-classification within the circular orders such that a curve and its inverse belong to the same sub-class.

Thus a non-singular bi-circular quartic can invert only into non-singular bi-circular quartics or circular cubics. The crunodal bi-circular quartics and circular cubics, with the hyperbolas, form another sub-class the members of which invert only one into another. The cuspidal bi-circular quartics and circular cubics, together with the parabola, form another class of curves bearing the same mutual relations, and finally the acnodal varieties of these higher curves are similarly associated with the ellipse. We have an example in the case of the cissoid of equation (7), which has a cusp at the point $(-a, 0)$; this cusp inverts into a cusp, which, since the modulus employed was a^2, is situated at the same point. Accordingly, if we transfer the origin in eq. (8) to the point $(-a, 0)$, we have the equation

$$r^4 - ar^2x - 2a^2y^2 = 0,$$

showing that the curve has a cusp at which the tangent is the axis of x. If we invert this curve with respect to its present origin, the cusp, we shall have the parabola

$$a^2 - ax - 2y^2 = 0.$$

DETERMINATION OF THE LOCUS OF O. (SEE FIG. ON P.22.)

BY CHRISTINE LADD, UNION SPRINGS, N. Y.

THE perpendicular on γ from $\alpha'\beta'\gamma'$ is

$$\alpha\beta' - \beta\alpha' = (\beta\gamma' - \gamma\beta')\cos B + (\gamma\alpha' - \alpha\gamma')\cos A,$$

and the line joining its intersection with γ to the vertex C is

$$\alpha(\beta' + \gamma'\cos A) - \beta(\alpha' + \gamma'\cos B) = 0.$$

The condition that the three lines of this kind meet in a point is

$$(\alpha' + \beta'\cos C)(\beta' + \gamma'\cos A)(\gamma' + \alpha'\cos B)$$
$$= (\alpha'\cos C + \beta')(\beta'\cos A + \gamma')(\gamma'\cos B + \alpha'),$$